Bounded Market Makers from Convex Duality: Self-Set Fees, Dead Bands, and the Order Book as a Theorem

Peter Cotton
Microprediction
peter.cotton@microprediction.com

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Abstract

We study automated market makers for contracts that settle to bounded real payoffs, such as a saturated surprise relative to a published forecast, rather than to the indicator of an outcome. The cost function $C(q) = b\sum_d \log\cosh(\langle q, v_d\rangle/b)$, built from *residual directions* v_d supplied by the market maker and its capital $b > 0$, has a convex conjugate that is finite exactly on the zonotope $V[-1,1]^D$. The worst-case loss is therefore exactly $b\,D\log 2$, the price set is the interior of the zonotope, and the Hessian is a covariance quoted by vectors, revealed through cross-impact and depleted by inventory. Any cost that is a state function refunds itself over a round trip, so revenue must be path-dependent, and a per-unit fee $f\|\delta\|_1$ set by each maker is a *dead band* in price space: a trader with belief π transacts nothing exactly when $\pi \in p(q) + [-f, f]^n$, and otherwise solves a lasso whose solution is sparse across contracts. Aggregating fee-bearing makers is an inf-convolution whose supply is the horizontal sum of band-limited supplies, so a consolidated limit order book, with a level at each maker's quote, depth equal to its curvature and exact zero fills for uncompetitive makers, follows as a theorem.

1 Introduction

A cost-function market maker [1, 7, 13, 14] quotes a price vector $p(q) = \nabla C(q)$ for a convex potential C of its net position q, and charges a trader who moves the position from q to $q + \delta$ the amount $C(q+\delta) - C(q)$. The logarithmic market scoring rule (LMSR) is the case $C(q) = b\log\sum_i e^{q_i/b}$ on a complete set of mutually exclusive outcomes, and its celebrated properties all follow from the convex-analytic structure of C [1]: prices that are probabilities, a worst-case loss of $b\log n$, and an interpretation as a sequentially shared proper scoring rule.

We work out the same programme for the setting that arises when a market is attached to a *forecast*: contracts that settle to a bounded, real-valued *surprise* relative to a published baseline rather than to the indicator of an outcome. Three design pressures shape the construction.

1. **Bounded loss must be exact, not asymptotic.** A market maker that posts capital b should be able to state its worst-case loss in closed form, and the bound should be attained at a nameable settlement. We obtain $b\,D\log 2$ for D quoted directions, attained at vertices of a zonotope (Theorem 3.3).

2. **Correlation must be quoted cheaply and truthfully.** A market maker across n related contracts has a view about their joint behaviour. Asking for an $n \times n$ covariance matrix is

both expensive and unreliable in practice (it is never positive semidefinite after editing). We ask instead for residual *vectors*; their Gram matrix is automatically PSD, and it is exactly the Hessian of C at the origin (Section 6). The cross-impact traders observe is therefore the covariance the maker is prepared to accept fills under.

3. **The protocol should fix as little as possible.** Fees are not a parameter of the mechanism but a quote of each maker; aggregation across makers should need no auction, no central book and no parameter. We show that with fee-bearing costs, convex duality alone produces the consolidated limit order book (Theorem 5.1).

The last point inverts the usual relationship between automated market makers and order books. Order books are typically presented as the primitive and automated makers as an approximation to them; here a book with discrete levels and finite depth is the *aggregate* of smooth makers, and the discreteness comes entirely from the fee.

The mathematics is elementary (inf-convolution, conjugacy and soft thresholding) and has a textbook economic reading: price is marginal cost, aggregation is Marshall's horizontal summation of supply curves, the conjugate is Hotelling's profit function. The two departures from producer theory are that production here is reversible, which is why the fee must be added by hand, and that the firm is a multiproduct one with economies of scope made tractable by compression of its cost structure into a few residual directions.

Contributions. (i) The bounded cost family and its conjugate in closed form, with an exact and attained loss bound and an exact description of the price set (Section 3). (ii) The fee as the unique path-dependent charge and as a dead band; the trader's problem as a lasso with exact sparsity (Section 4). (iii) The aggregation theorem: the conjugate of the inf-convolution is the sum of the conjugates; the aggregate supply is the horizontal sum of band-limited supplies; the limit order book, the additivity of liquidity near the origin and the one-dimensional routing problem (Section 5). (iv) The curvature structure: covariance quoted by vectors, revealed by cross-impact, depleted by inventory, and a conservation law for curvature and collateral (Section 6). (v) The duality with proper scoring rules and follow-the-regularised-leader, and the fee as a regularisation parameter (Section 7). (vi) Numerical verification of every statement (Section 8 and Appendix A).

2 Setting

There are n contracts. Contract i settles to a real number $s_i \in [-1, 1]$. In the application that motivates us, $s_i = \tanh(z_i/\kappa)$ where z_i is a standardised surprise, the realised value of a quantity minus a frozen baseline forecast, in units of the baseline's own standard deviation. The saturating transform is deliberate: it bounds payouts, makes the influence of a single corrupt observation bounded, guarantees that all moments exist, and makes the price a bounded quantity, the market's expected correction to the baseline. Nothing below depends on the choice of tanh; only $s \in \mathcal{S} \subseteq [-1, 1]^n$ is used.

A market maker holds a net position $q \in \mathbb{R}^n$: q_i is the number of units of contract i it has sold. It quotes through a convex, differentiable cost function $C : \mathbb{R}^n \to \mathbb{R}$ with $C(0) = 0$. A trader who buys the bundle $\delta \in \mathbb{R}^n$ pays $C(q + \delta) - C(q)$, and the instantaneous price is $p(q) = \nabla C(q)$. At settlement the maker pays $\langle q, s \rangle$. Starting from $q = 0$, the maker's realised profit after a sequence of trades totalling q is $C(q) - \langle q, s \rangle$; it does not depend on the path, only on q. We

write $C^*(s) = \sup_q\{\langle q, s\rangle - C(q)\}$ for the convex conjugate, so that the maker's worst-case loss against settlement s is exactly $C^*(s)$.

3 The bounded cost family

Definition 3.1 (Residual directions and capital)**.** Let $V = [v_1 \ \cdots \ v_D] \in \mathbb{R}^{n\times D}$ have full row rank, let $b > 0$, and let $\ell(x) = \log\cosh x$. The *bounded market maker* with directions V and capital b has cost

$$C_{b,V}(q) = b\sum_{d=1}^{D} \ell\!\left(\frac{\langle q, v_d\rangle}{b}\right) = b\,\mathbf{1}^\top \ell\Big(V^\top q/b\Big), \tag{1}$$

price $p(q) = \nabla C_{b,V}(q) = V\tanh(V^\top q/b)$, and Hessian

$$H(q) = \nabla^2 C_{b,V}(q) = \tfrac{1}{b}\,V\,\mathrm{diag}\big(\operatorname{sech}^2(V^\top q/b)\big)V^\top. \tag{2}$$

The function ℓ is even, convex, $\ell(0) = \ell'(0) = 0$, $\ell''(x) = \operatorname{sech}^2 x \le 1$, and $\ell(x) = |x| - \log 2 + o(1)$ as $|x| \to \infty$. Its conjugate is the binary-entropy-like function

$$\ell^*(y) = \sup_x\{xy - \ell(x)\} = \begin{cases} \tfrac{1}{2}\big[(1+y)\log(1+y) + (1-y)\log(1-y)\big], & |y| \le 1, \\ +\infty, & |y| > 1, \end{cases} \tag{3}$$

with $\ell^*(0) = 0$, $\ell^*(\pm 1) = \log 2$, $(\ell^*)'(y) = \operatorname{artanh} y$, and $0 \le \ell^* \le \log 2$ on $[-1, 1]$.

Lemma 3.2 (Conjugate)**.** *Let $\mathcal{Z}_V = V[-1,1]^D = \{Vy : \|y\|_\infty \le 1\}$ be the zonotope generated by the columns of V. Then*

$$C^*_{b,V}(s) = \min\Big\{\, b\sum_{d=1}^{D} \ell^*(y_d) \ :\ Vy = s,\ \|y\|_\infty \le 1\Big\} \quad \text{for } s \in \mathcal{Z}_V, \tag{4}$$

and $C^*_{b,V}(s) = +\infty$ for $s \notin \mathcal{Z}_V$. In particular $0 \le C^*_{b,V} \le b\,D\log 2$ on $\mathcal{Z}_V$, with $C^*_{b,V}(s) = b\,D\log 2$ whenever $s = Vy$ for a vertex $y \in \{-1,1\}^D$ with no other representation in $[-1,1]^D$.

Proof. Write $C_{b,V} = g \circ V^\top$ with $g(z) = b\sum_d \ell(z_d/b)$, a finite convex function on $\mathbb{R}^D$ with $g^*(y) = b\sum_d \ell^*(y_d)$. Since g is finite everywhere and $V^\top$ is linear, Rockafellar [17, Thm. 16.3] gives $(g \circ V^\top)^*(s) = \inf\{g^*(y) : Vy = s\}$ and the infimum is attained when finite. $\operatorname{dom} g^* = [-1,1]^D$, so the conjugate is finite exactly on $V[-1,1]^D$. The bounds follow from $0 \le \ell^* \le \log 2$ on $[-1,1]$, and the vertex statement from the uniqueness of the representation. □

Theorem 3.3 (Exact worst-case loss)**.** *Let the settlement set satisfy $\mathcal{S} \subseteq \mathcal{Z}_V$. Then for every trading history the loss of the bounded market maker is at most $\sup_{s\in\mathcal{S}} C^*_{b,V}(s) \le b\,D\log 2$, and the bound $b\,D\log 2$ is attained: if $s = Vy$ with $y \in \{-1,1\}^D$ a uniquely represented vertex, the trader bundle $q_t = t\,V^{-\top}y$ with $t \to \infty$ (when $D = n$, so that every y is uniquely represented) drives the loss to $b\,D\log 2$. If $\mathcal{S} \not\subseteq \mathcal{Z}_V$ the worst-case loss is infinite.*

Proof. The maker's loss against s after trades totalling q is $\langle q, s\rangle - C(q) \le C^*(s)$, with the supremum over q equal to $C^*(s)$ by definition; apply Lemma 3.2. For attainment, $V^\top q_t = ty$, so $\langle q_t, s\rangle - C(q_t) = t\|y\|^2 - b\sum_d \ell(t/b) \to b\,D\log 2$ since $\|y\|^2 = D$ and $t - b\ell(t/b) \to b\log 2$. If some $s \in \mathcal{S}$ lies outside $\mathcal{Z}_V$ then $C^*(s) = +\infty$ and the supremum over q is unbounded. □

Corollary 3.4 (Defaults guarantee the bound)**.** *If V contains the scaled coordinate directions $m_i e_i$ with $m_i \geq 1$ for every i (the maker always quotes at least the baseline's own marginals, possibly inflated by a volatility multiplier) then $[-1,1]^n \subseteq \mathcal{Z}_V$, so every settlement in $[-1,1]^n$ is covered and Theorem 3.3 applies with $D \geq n$. Adding further residual directions only enlarges $\mathcal{Z}_V$ and raises the bound by $b \log 2$ per direction.*

Proposition 3.5 (The price set)**.** *$p(\mathbb{R}^n) \subseteq \operatorname{int} \mathcal{Z}_V$. If $D = n$ (so V is invertible) then $p(\mathbb{R}^n) = \operatorname{int} \mathcal{Z}_V = V(-1,1)^n$, and p is a diffeomorphism of $\mathbb{R}^n$ onto it with inverse $q(p) = b\, V^{-\top} \operatorname{artanh}(V^{-1}p)$.*

Proof. $p(q) = V \tanh(V^\top q/b)$ and tanh maps $\mathbb{R}^D$ into $(-1,1)^D$, whose image under V is $\operatorname{int} \mathcal{Z}_V$ when V has full row rank. When $D = n$, $V^\top q$ ranges over all of $\mathbb{R}^n$ and tanh is a bijection onto $(-1,1)^n$; the inverse formula is immediate. □

Proposition 3.5 is the analogue of "LMSR prices are probabilities": prices lie in the convex hull of the settlement set, so no portfolio has a riskless profit against the maker, and the maker never quotes a price it could not be settled at. The zonotope replaces the simplex because settlements are bounded reals rather than indicators of exhaustive outcomes.

Example 3.6 (One contract). With $n = D = 1$, $v = 1$: $C(q) = b \log \cosh(q/b)$, $p(q) = \tanh(q/b)$, $q(p) = b \operatorname{artanh} p$, $C^*(s) = b\ell^*(s)$, worst-case loss $b \log 2$ at $s = \pm 1$. The loss $|q| - C(q)$ as a function of the position is increasing and saturates at $b \log 2$; the price saturates at ± 1; the depth $1/p'(q) = b \cosh^2(q/b)$ grows without bound as the maker's inventory grows, so the maker charges more and more per unit as it approaches its loss bound, and would require an infinite payment to move its price to ± 1. Compare the quadratic cost $q^2/2b$, which is Gaussian at the core but has unbounded loss, and the LMSR, which is bounded but whose price moves with the same curvature at every inventory level. The log-cosh maker is Gaussian at the core and bounded at the edges.

4 Fees and the dead band

State functions refund themselves. For any cost C, a round trip $q \to q + \delta \to q$ costs the trader $[C(q+\delta) - C(q)] + [C(q) - C(q+\delta)] = 0$. This is not a defect of a particular C but of the class: a charge that is a state function along every path integrates to zero around every closed loop. Curvature therefore cannot be a source of revenue; it is an inventory-control device. The market maker's revenue, and its only defence against a trader who repeatedly round-trips to extract information or liquidity, must be path-dependent.

Definition 4.1 (Fee-bearing cost)**.** Fix a fee $f \in \mathbb{R}^n_{\geq 0}$ (per unit, per contract). At state q the trader buying δ pays

$$C_q^f(\delta) = C(q+\delta) - C(q) + \langle f, |\delta| \rangle, \tag{5}$$

where $|\delta|$ is componentwise. A round trip of size δ now costs $2\langle f, |\delta| \rangle > 0$.

Among charges that are positively homogeneous of degree one in δ, continuous, and depend on δ alone, the weighted ℓ_1 norm is the natural one: it is the support function of the box $[-f, f]^n$, so it is the cheapest path-dependent charge whose conjugate is an indicator, and that is exactly what makes the fee a dead band.

Proposition 4.2 (The fee is a dead band in price space)**.** *Let a trader hold a belief $\pi \in \mathbb{R}^n$ about the settlement mean and choose δ to maximise $\langle \delta, \pi \rangle - C_q^f(\delta)$. Then*

1. *$\delta = 0$ is optimal if and only if $\pi \in p(q) + [-f, f]^n$, i.e. $|\pi_i - p_i(q)| \le f_i$ for every i. Thus the maker's bid and ask for contract i are $p_i(q) \mp f_i$, and the bid–ask spread is $2f_i$.*
2. *At an optimum, for every i with $\delta_i \neq 0$, $p_i(q+\delta) = \pi_i - f_i \operatorname{sgn} \delta_i$, and for every i with $\delta_i = 0$, $|\pi_i - p_i(q+\delta)| \le f_i$.*
3. *The conjugate of the fee-bearing cost is an inf-convolution: $(C_q^f)^*(\pi) = \inf_{\|u\|_\infty \le f} \{C^*(\pi - u) - \langle q, \pi - u \rangle + C(q)\}$, the ℓ_∞-ball of radius f smearing the conjugate of the fee-free cost. When $C(q) = \sum_i C_i(q_i)$ is separable, the optimal trade moves each price to the soft-thresholded belief, $p_i(q+\delta) = p_i(q) + \mathrm{ST}_{f_i}\big(\pi_i - p_i(q)\big)$ with $\mathrm{ST}_f(x) = \operatorname{sgn}(x) \max(|x| - f, 0)$.*

Proof. The objective is concave in δ; 0 is optimal iff $\pi \in \partial C_q^f(0) = \nabla C(q) + \partial \langle f, |\cdot| \rangle(0) = p(q) + [-f, f]^n$. Item 2 is the subgradient condition $\pi \in \nabla C(q+\delta) + f \odot \partial|\delta|$. Item 3 is the conjugate of a sum, $(A+B)^* = A^* \,\square\, B^*$ [17, Thm. 16.4], with $B = \langle f, |\cdot| \rangle$ whose conjugate is the indicator of $\|\cdot\|_\infty \le f$ componentwise, and $A(\delta) = C(q+\delta) - C(q)$ whose conjugate is $C^*(\cdot) - \langle q, \cdot \rangle + C(q)$. In the separable case item 2 reads $p_i(q_i + \delta_i) = \pi_i - f_i \operatorname{sgn} \delta_i$ off the band and $\delta_i = 0$ on it, which is the soft-threshold formula. □

For a single contract, the fee–spread duality for any convex cost and the exact no-trade interval for any cost whatever are Lemmas 4 and 5 of Cotton [9]; the statement above is the multi-contract form, with the ℓ_∞-ball of item 3 in place of the scalar interval.

Item 2 is a lasso. The trader's problem $\max_\delta \langle \delta, \pi \rangle - [C(q+\delta) - C(q)] - \langle f, |\delta| \rangle$ has an ℓ_1 penalty, so its solution is *exactly* sparse: contracts whose price lies inside the band are not touched at all, however small the fee. This is the mechanism behind the sparse routing of Section 5.

Fees are self-set. Nothing in the protocol fixes f. A maker who sets f too low is picked off by informed flow it cannot refuse; one who sets it too high transacts nothing and earns nothing. The fee is thus a quote in the ordinary sense: a price for immediacy, set by competition, and its level is the maker's own estimate of adverse selection in the sense of Glosten and Milgrom [11]. We need no external market in volatility or in fees to determine it, and a mis-set fee punishes its setter and nobody else.

5 The order book as a theorem

Let there be K market makers with costs C_k, states q_k and fees f_k. A trade δ arriving at the venue is split as $\delta = \sum_k \delta_k$ to minimise the trader's total cost,

$$\mathcal{C}(\delta) = \min \Big\{ \sum_{k=1}^{K} C_{q_k}^{f_k}(\delta_k) \;:\; \sum_k \delta_k = \delta \Big\} = \big(C_{q_1}^{f_1} \,\square\, \cdots \,\square\, C_{q_K}^{f_K}\big)(\delta), \tag{6}$$

the inf-convolution of the fee-bearing costs.

Theorem 5.1 (Aggregation)**.** *1.* ***Conjugates add.*** *$\mathcal{C}^* = \sum_k (C_{q_k}^{f_k})^*$. In particular, for fee-free bounded makers starting at the origin, the aggregate worst-case loss against s is $\sum_k C_k^*(s)$, and the aggregate loss bound is $\sum_k b_k D_k \log 2$.*

2. ***One clearing price.*** *At an optimal split there is a single vector π such that for every maker k and contract i: if $\delta_{k,i} \neq 0$ then $p_{k,i}(q_k + \delta_k) = \pi_i - f_{k,i} \operatorname{sgn} \delta_{k,i}$, and if $\delta_{k,i} = 0$ then $|\pi_i - p_{k,i}(q_k + \delta_k)| \leq f_{k,i}$. Every maker that transacts in contract i does so at the same marginal price π_i net of its own fee.*

3. ***Supplies add horizontally; bands give exact sparsity.*** *Let $S_k(\pi)$ be maker k's supply correspondence, the set of δ_k satisfying item 2 at π. The aggregate supply is $S(\pi) = \sum_k S_k(\pi)$ (Minkowski sum), and $S_k(\pi) = \{0\}$ whenever $\pi \in p_k(q_k) + [-f_k, f_k]^n$. A maker whose band contains the clearing price fills exactly zero.*

4. ***Liquidity adds near the origin.*** *If all fees are zero and all makers are at the origin, then $\nabla^2 \mathcal{C}(0)^{-1} = \sum_k H_k(0)^{-1}$: inverse Hessians (liquidities) add like conductances in parallel.*

5. ***One contract is a monotone root-find.*** *For $n = 1$ with bounded makers $C_k(q) = b_k \log \cosh(q/b_k)$ at prices $p_k = \tanh(q_k/b_k)$, maker k's supply at a clearing price π is the closed form*

$$S_k(\pi) = \begin{cases} b_k \operatorname{artanh}(\pi - f_k) - q_k, & \pi > p_k + f_k, \\ 0, & |\pi - p_k| \leq f_k, \\ b_k \operatorname{artanh}(\pi + f_k) - q_k, & \pi < p_k - f_k, \end{cases} \tag{7}$$

which is continuous and nondecreasing in π on $(-1 + f_k, 1 - f_k)$; the aggregate $S(\pi) = \sum_k S_k(\pi)$ is continuous and strictly increasing wherever at least one maker is active, so the clearing price for a trade of size δ is the unique root of $S(\pi) = \delta$, found by bisection. The depth function $S'(\pi) = \sum_{k \text{ active}} b_k / \big(1 - (\pi \mp f_k)^2\big)$ is a ladder: each maker enters at its own quote with depth equal to its curvature.

Proof. (1) is the conjugate of an inf-convolution [17, Thm. 16.4]; the fee-free statement follows from Lemma 3.2. (2) The Lagrangian of (6) is $\sum_k C_{q_k}^{f_k}(\delta_k) - \langle \pi, \sum_k \delta_k - \delta \rangle$; the multiplier π is common to all k, and stationarity in δ_k is item 2 of Proposition 4.2 with π in place of the trader's belief. (3) The supply correspondence is $\partial (C_{q_k}^{f_k})^*(\pi)$, and subdifferentials of a sum of convex functions with a common point of continuity add; the zero statement is Proposition 4.2(1). (4) Fee-free costs at the origin have $\mathcal{C}^* = \sum_k C_k^*$, so $\nabla^2 \mathcal{C}^*(0) = \sum_k \nabla^2 C_k^*(0) = \sum_k H_k(0)^{-1}$ by the inverse-function theorem applied to ∇C_k at the origin; invert once more. (5) Solve $\tanh((q_k + \delta_k)/b_k) = \pi \mp f_k$ for δ_k; monotonicity and continuity follow from those of artanh and the matching of the three branches at the band edges; the derivative is $\frac{d}{d\pi} b_k \operatorname{artanh}(\pi \mp f_k)$. □

For one contract and general convex costs, items 1 to 3 are Lemma 6 and Corollary 8 of Cotton [9], where the dead zones and the walk through the book are established without the bounded family. Items 4 and 5, the addition of inverse Hessians and the closed-form supply curves, use the bounded family.

The consolidated book. Item 2 is the law of one price: a trade clears at a single marginal price per contract, and every maker that participates is indifferent at the margin. Item 3 says that a maker's quote is literally a level in a book: below its bid and above its ask it is absent, and between them it is present with the depth given by its curvature. Item 5 makes this concrete for a single contract: the aggregate depth is a ladder with a rung at each maker's bid and ask. Large trades therefore "eat through" makers in order of their quotes, exactly as a market order walks a book, and small trades go entirely to the single cheapest maker because the others' bands have not been crossed. Item 4 says that in the small-trade regime the venue behaves like one

maker whose liquidity is the sum of the participants' liquidities, which is Marshall's horizontal summation of supply curves: inverse Hessians add. Saturation, which is where the makers differ from a Gaussian, only appears near each maker's capital scale.

Routing cost. Item 5 is a one-dimensional bisection with closed-form evaluations, so routing a trade across K makers costs $O(K \log(1/\varepsilon))$ evaluations of artanh. For $n > 1$ the split is a smooth convex programme with ℓ_1 terms; it is small (the active set is sparse by item 3) and a few Newton steps on the active set, each requiring the solution of a system whose matrix is a sum of the active makers' Hessians, suffice.

6 Curvature as a quoted covariance

Quoted by vectors. At the origin, $H(0) = VV^\top/b$. A maker who supplies residual vectors $v_1, \ldots, v_D$ has thereby quoted a covariance $VV^\top$, up to the scale b: the Gram matrix of the residual vectors, PSD by construction, with no eigenvalue bookkeeping. If each v_d is expressed relative to the baseline's own marginal scales, $\|v_d\|$ is a volatility view (1.0 means "defer to the baseline", 1.3 means "volatility runs 30% hot") and the direction of v_d is a correlation view. The quote is robust to recalibration of the baseline, since it is relative, and it is exactly what a joint forecast looks like when written as a set of residual scenarios: the maker is a forecaster paid by the spread rather than by a score.

Revealed by cross-impact. A trader who buys δ observes the price move $p(q+\delta) - p(q) \approx H(q)\delta$. The off-diagonal of H is the cross-impact matrix, and at the origin it is the quoted covariance. Nothing in the protocol asks the maker to publish V; the covariance is revealed by the maker's willingness to let its price in contract j move when contract i is traded. The quote is incentive-compatible in the following sense: the maker accepts fills at prices that move according to H, so quoting a covariance it does not believe means volunteering for positions it considers mispriced. Means are pinned by settlement and enforced by arbitrage; covariances are never settled, but they are revealed by the cross-impact of the mean markets as a by-product of risk management rather than as a separate product. When a particular pair's covariance is wanted as a settled quantity, a contract on $z_i z_j$ promotes it to first class.

Depleted by inventory. Away from the origin, $H(q) = \frac{1}{b}\sum_d \operatorname{sech}^2(\langle q, v_d\rangle/b)\, v_d v_d^\top$. As the maker accumulates inventory along direction d, the factor sech^2 falls and the curvature in that direction flattens: the maker's price in a loaded direction becomes insensitive to further flow because it is approaching the boundary of the zonotope, where it has promised its entire capital. Prices remain right in the sense of Proposition 3.5, but the *indicated* covariance of an aged book is inventory-contaminated: cross-impact equals beliefs times remaining capacity. Two consequences follow. First, covariance should be read off fresh books, those at or near $q = 0$, never aged ones. Second, the tempting fix of re-centring q at the current inventory so that curvature is restored breaks Theorem 3.3: each re-centring re-arms $b\,D\log 2$ of potential loss without any new collateral.

Proposition 6.1 (Conservation of curvature and collateral)**.** *Let a bounded maker with capital b be re-centred m times (its position reset to zero for pricing purposes while its settlement obligations are retained). Its worst-case loss is $(m+1)\,b\,D\log 2$. Curvature in a direction can be restored*

without raising the loss bound only by settlement of the contracts, by unwinding the position through other makers at their fees, or by committing new capital b', in which case the bound rises by exactly $b'D\log 2$.

Proof. Each re-centred epoch is a fresh instance of Theorem 3.3 with the same b, and the settlement obligations of the epochs add. Settlement discharges the obligation; unwinding transfers it; new capital raises the bound by the bound of the new maker. □

Saturation in a direction is therefore not a problem for the protocol to solve but a signal for the market to act on: a wide marginal price in some direction is an invitation to a new maker, and entry restores curvature.

7 Duality with scoring rules and online learning

The correspondence between cost functions and proper scoring rules [1, 7, 13] holds here verbatim. If $p = \nabla C(q)$ and C is strictly convex with C^* differentiable at p, then $q = \nabla C^*(p)$ and

$$\langle q, s\rangle - C(q) = C^*(p) + \langle \nabla C^*(p), s - p\rangle =: \mathsf{S}_{C^*}(p; s), \tag{8}$$

so the maker's loss against settlement s is the proper scoring rule generated by the convex function C^*, evaluated at its own quoted mean p. For the bounded maker in one dimension $C^*(p) = b\ell^*(p)$ and $\mathsf{S}(p; s) = b[\ell^*(p) + \operatorname{artanh}(p)(s - p)]$, a bounded, proper score for a mean in $[-1, 1]$ whose maximum, $b\log 2$, is the loss bound. The maker is a forecaster who publishes a mean and is scored by S, with the difference that it is paid in fees rather than subsidised, and that any trader may move its forecast at a price.

Abernethy et al. [1] also show that a cost-function maker's price path is the follow-the-regularised-leader algorithm with regulariser C^*: $p_t = \arg\max_p\{\langle p, q_t\rangle - C^*(p)\}$ with q_t the cumulative flow. For the bounded maker the regulariser is $b\sum_d \ell^*(y_d)$ in direction coordinates, so prices are logistic-type functions of cumulative flow per direction, and the learning rate is $1/b$: capital is the inverse learning rate, a point also visible in Theorem 5.1(4).

The fee is a regularisation parameter. Proposition 4.2(2) exhibited the trader's problem as a lasso with penalty f. The aggregation of Theorem 5.1 exhibits the *venue* as a lasso over makers: at clearing price π the active set is the set of makers whose quote clears their own fee. Lowering a common fee f activates makers in order of their edge, tracing a regularisation path in the manner of least-angle regression [10]: the order book under a falling fee is the LARS path of the ensemble of makers. In this reading a fee-free venue is ordinary least squares over makers (every maker participates, liquidities add, Theorem 5.1(4)), a fee is the ℓ_1 term that produces exact sparsity, and a maker's capital b is its shrinkage. We do not develop this further here; it is a precise form of the folk observation that market selection among forecasters is a form of regularised estimation [6, 8].

8 Numerical illustration

Figures 1–3 are produced by the script of Appendix A, which also runs the numerical checks listed there.

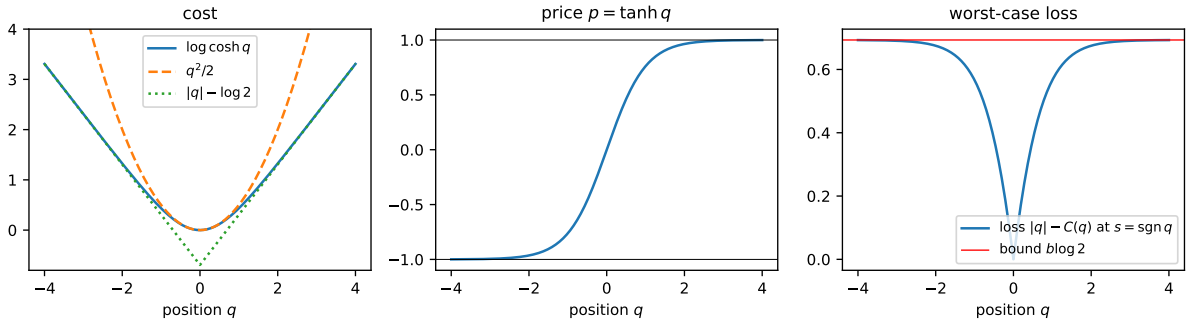


Figure 1: One contract, $b = 1$. Left: cost $C(q) = \log\cosh q$ against the quadratic $q^2/2$ and the asymptote $|q| - \log 2$. Middle: price $p(q) = \tanh q$, saturating at ± 1. Right: the realised loss $|q| - C(q)$ against the worst settlement $s = \operatorname{sgn} q$, which approaches but never exceeds the exact bound $\log 2$ (Theorem 3.3).

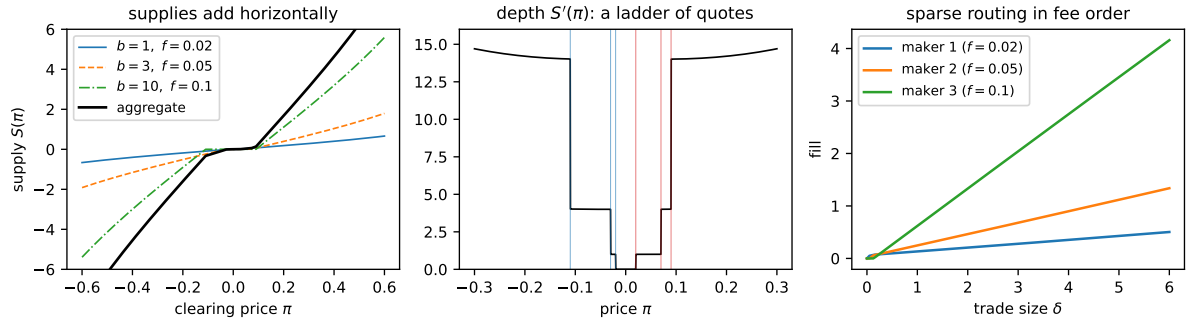


Figure 2: Three makers on one contract, with capital, fee and current price (b_k, f_k, p_k) equal to $(1, 0.02, 0)$, $(3, 0.05, 0.02)$ and $(10, 0.10, -0.01)$. Left: each maker's supply (7) is zero inside its own band and the aggregate is their horizontal sum (Theorem 5.1). Middle: the depth $S'(\pi)$ near the quotes is a ladder with a rung at every bid (blue) and ask (red); this is the consolidated book. Right: the fill allocation of a trade of size δ across the three makers: small trades go entirely to the cheapest maker and larger trades eat through the others in fee order, with exact zeros.

9 Related work

Cost-function market makers originate with Hanson's market scoring rules [13, 14]; the utility-based view is Chen and Pennock [7]; the convex-analytic framework, including the characterisation of bounded loss through the conjugate and the identification of price dynamics with follow-the-regularised-leader, is Abernethy et al. [1]. The scalar fee–spread duality, the exact no-trade interval for non-convex costs, and the order book as the horizontal sum of fee-shifted supplies are developed in Cotton [9]. The many-contract form and the closed-form family extend them. Liquidity-sensitive and profit-charging variants of LMSR are studied by Othman et al. [16]; our fee differs in that it is per unit and ℓ_1, which produces exact sparsity and a dead band rather than a continuously widening spread.

Bounded settlements of the tanh form are standard in robust statistics as bounded-influence transforms [15]. The dead-band view of the spread is the adverse-selection view of Glosten and

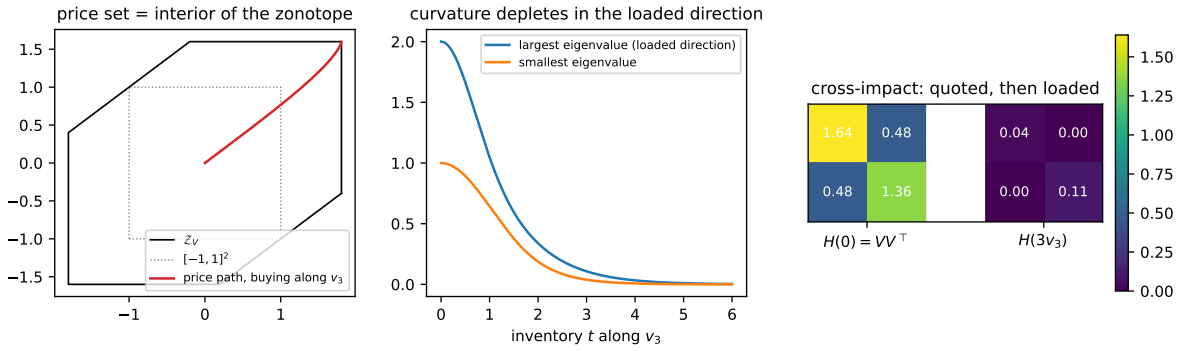


Figure 3: Two contracts, $V = [v_1\, v_2\, v_3]$ with $v_1 = e_1,\ v_2 = e_2,\ v_3 = (0.8, 0.6)$, $b = 1$. Left: the zonotope $\mathcal{Z}_V = V[-1,1]^3$ (the price set and the set of settlements with finite worst-case loss) with the price path of a trader buying along v_3. Middle: the eigenvalues of the Hessian $H(q)$ along that path; the loaded direction flattens as sech^2 falls while the orthogonal direction keeps its curvature. Right: the cross-impact matrix at the origin (the quoted covariance $VV^\top$) and after loading.

Milgrom [11]; inventory-based spreads in the style of Avellaneda and Stoikov [4] are complementary and can be read off the Hessian in Section 6.

Inf-convolution and the additivity of conjugates are classical [17]; their use for aggregating supply is Marshall's, and the multiproduct cost reading is Baumol et al. [5]. Constant-function market makers in decentralised finance [2] are also convex-duality objects (trading sets and their support functions), and their aggregation across pools by routing [3] is an inf-convolution of the same kind, though without the fee-as-band structure and with reserves in place of a loss bound.

Economic-derivatives auctions on macroeconomic releases [12] are the closest historical precedent for markets on standardised surprises; they were episodic parimutuel auctions without shared-state makers. The identification of market selection with Bayesian averaging is Blume and Easley [6]; cost-function makers as online learners is Chen and Vaughan [8].

10 Conclusion

A market maker that posts two numbers, a fee and a capital, and optionally a few residual vectors, has stated a bounded loss exactly, quoted a covariance it accepts fills under, and set its own spread; a set of such makers aggregates, by convex duality alone, into a consolidated limit order book with sparse routing and additive liquidity. The protocol needs to fix only the settlement rule. Open questions remain about the dynamics: the lifecycle of such markets as the baseline freezes and the observation approaches, the toxicity of near-settlement flow, and the equilibrium fees that a population of competing makers settles on. Those are empirical questions.

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A Numerical verification

The script `verify.py`, distributed with the source, checks the following with the stated tolerances and prints a pass/fail line for each; all pass.

1. **Conjugate formula** (Lemma 3.2). For $V = I_3$, $b \in \{0.5, 1, 3\}$ and 200 random $s \in (-1, 1)^3$, $\sup_q\{\langle q, s\rangle - C(q)\}$ by numerical maximisation agrees with $b\sum_i \ell^*(s_i)$ to 10^{-6}. For a non-square V ($n = 2$, $D = 3$) the numerical supremum agrees with the constrained minimum in (4) to 10^{-5}.
2. **Loss bound and attainment** (Theorem 3.3). For 2000 random trade sequences and every vertex settlement, loss $\le b\,D\log 2 + 10^{-9}$; along $q_t = tV(V^\top V)^{-1}y$ the loss exceeds $b\,D\log 2 - 10^{-6}$ for large t. For a settlement outside $\mathcal{Z}_V$ the loss along a ray grows linearly.
3. **Price set** (Proposition 3.5). For $D = n$ and random q, $V^{-1}p(q) \in (-1, 1)^n$, and $q(p(q)) = q$ to 10^{-9}.
4. **Dead band** (Proposition 4.2). For random q, π, f the maximiser of $\langle\delta, \pi\rangle - C_q^f(\delta)$ (found by a proximal-gradient iteration to 10^{-10}) is zero iff $\|\pi - p(q)\|_\infty \le f$, and the active-set conditions of item 2 hold to 10^{-7}.
5. **Conjugates add** (Theorem 5.1(1)). For two one-dimensional fee-bearing makers, the conjugate of the inf-convolution computed by brute force on a grid equals the sum of the conjugates to 10^{-4} (the grid resolution).
6. **One clearing price and horizontal summation** (Theorem 5.1(2,3,5)). For three makers and 200 random trade sizes, the optimal split found by direct minimisation agrees with the bisection on (7) to 10^{-7} in each fill; inactive makers have fills exactly zero; the Lagrange multiplier equals the clearing price.
7. **Liquidity adds** (Theorem 5.1(4)). For three fee-free makers at the origin the numerical Hessian of the inf-convolution at 0 has inverse equal to $\sum_k H_k(0)^{-1}$ to 10^{-5}.
8. **Bregman identity** (equation (8)) to 10^{-9} for random q, s.
9. **Re-centring** (Proposition 6.1). A maker re-centred m times attains loss within 10^{-4} of $(m + 1)b\log 2$ in one dimension.